PERIODIC SOLUTIONS OF QUASILINEAR AUTONOMOUS SYSTEMS WHICH HAVE FIRST INTEGRALS

(PERIODICHESKIE RESHENIIA KVAZILINEINYKH AVTONOMNYKH SISTEM, OBLADAIUSHCHIKH PERVYMI INTEGRALAMI)

PMM Vol.27, No.2, 1963, pp. 369-372

Iu. A. ARKHANGEL'SKII
(Moscow)

(Received November 1, 1962)

1. We consider a quasilinear autonomous system with n degrees of freedom

$$\sum_{k=1}^{n} (a_{ik}\ddot{x}_k + c_{ik}x_k) = \mu f_i (x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n, \mu) \qquad (i = 1, \ldots, n) \qquad (1.1)$$

where the functions f_i are analytic functions of their arguments in some region of variation, μ is a small parameter and all the roots of the frequency equation

$$\Delta(\omega^2) = | c_{ik} - \omega^2 a_{ik} | = 0$$

are distinct and commensurable. Then the solution of the generating system

$$\sum_{k=1}^{n} (a_{ik}\ddot{x}_k + c_{ik}x_k) = 0 \quad (i = 1, \ldots, n), \qquad a_{ik} = a_{ki}, \qquad c_{ik} = c_{ki}$$

will contain all n frequencies $\omega_1, \ldots, \omega_n$ and will be periodic with some period T_0. We shall assume that the above solution of the generating system is associated with a periodic solution (1.1) having a period $T_0 + \alpha$ (α vanishes when $\mu = 0$); we shall represent the initial conditions corresponding to this periodic solution in the form [1, 2]

$$x_k(0) = \sum_{r=1}^{n} p_k^{(r)} (A_r + \beta_r), \qquad \dot{x}_k(0) = \sum_{r=2}^{n} p_k^{(r)} (B_r + \gamma_r)$$

Here A_r and B_r are constants; β_r and γ_r are functions of μ which vanish when $\mu = 0$, and

$$p_k^{(r)} = \frac{\Delta_{ik}(\omega_r^2)}{\Delta_{i1}(\omega_r^2)} \qquad (i = 1, \ldots, n)$$

where $\Delta_{ik}(\omega_r^2)$ is the cofactor of the element $c_{ik} - \omega_r^2 a_{jk}$ in the determinant $\Delta(\omega_r^2)$. In this case the expansion of the periodic solution of the system (1.1) in powers of the parameters β, γ and μ may be taken to have the following form [1,2]:

$$x_k(t) = (A_1 + \beta_1)\cos\omega_1 t + \sum_{r=2}^{n} p_k^{(r)}\left[(A_r + \beta_r)\cos\omega_r t + \frac{B_r + \gamma_r}{\omega_r}\sin\omega_r t\right] +$$
$$+ \sum_{m=1}^{\infty}\left[C_{km}(t) + \frac{\partial C_{km}}{\partial A_1}\beta_1 + \ldots + \frac{\partial C_{km}}{\partial B_n}\gamma_n + \ldots\right]\mu^m$$

where the expressions for $C_{km}(t)$ are given in the indicated references. Using the notation

$$x_k(T_0 + \alpha) - x_k(0) = \psi_k, \qquad \dot{x}_k(T_0 + \alpha) - \dot{x}_k(0) = \psi_{n+k} \qquad (k = 1, \ldots, n) \tag{1.2}$$

we obtain $2m$ periodicity conditions for the quantities $x_k(t)$ and $\dot{x}_k(t)$

$$\psi_m = 0 \qquad (m = 1, \ldots, 2n) \tag{1.3}$$

From these conditions we must determine not only the $2n - 1$ constants $A_1, \ldots, A_n, B_2, \ldots, B_n$, but also $2n$ functions of μ: $\beta_1, \ldots, \beta_n, \gamma_2, \ldots, \gamma_n, \alpha$ (since the system (1.1) is autonomous, $B_1 = \gamma_1 = 0$). One of these conditions, for example, $\psi_1 = 0$, will be used to determine the parameter α in the form of a series in integer powers of the β and γ values and μ

$$\alpha = \alpha(\beta_1, \ldots, \beta_n, \gamma_2, \ldots, \gamma_n, \mu) \tag{1.4}$$

This can always be done provided that

$$B_2 + \ldots + B_n \neq 0 \tag{1.5}$$

Expanding the left-hand sides of the remaining formulas of (1.2) in terms of α and substituting the expression (1.4) for α, we obtain in the general case the following equations:

$$\mu^{s_j}\left[M_j(A_1, \ldots, A_n, B_2, \ldots, B_n) + N_j(\beta_1, \ldots, \beta_n, \gamma_2, \ldots, \gamma_n, \mu)\right] = \psi_j$$
$$(j = 2, \ldots, 2n) \tag{1.6}$$

where s_j are positive integers not less than unity, M_j are functions of the constants A and B, and the expressions N_j are analytic functions of all their arguments in a neighborhood of their zero values, with $N_j(0, \ldots, 0) = 0$. It can also be shown (in a manner similar to the proof for $n = 2$ given in [3]) that

$$N_j = \frac{\partial M_j}{\partial A_1}\beta_1 + \ldots + \frac{\partial M_j}{\partial A_n}\beta_n + \frac{\partial M_j}{\partial B_2}\gamma_2 + \ldots + \frac{\partial M_j}{\partial B_n}\gamma_n + \ldots + \mu(\ldots) \quad (1.7)$$

The same study [3] calculates, for $n = 2$, the functions M_j and the coefficients of the first three terms of the expansions of the functions N_j as power series in μ for $s_j = 1$.

Thus, the periodicity conditions (1.3) for the system (1.6) may be divided into two groups of conditions

$$(1) \quad M_j = 0, \qquad (2) \quad N_j = 0 \qquad (j = 2, \ldots, 2n) \quad (1.8)$$

From the first group of conditions, called the equations of basic amplitudes, we find the constants A and B, and from the second group of conditions we find the functions $\beta(\mu)$ and $\gamma(\mu)$. We then substitute the resulting initial values A, B, $\beta(\mu)$ and $\gamma(\mu)$ of the desired periodic solution into formula (1.4) to find the correction value for α per period.

2. Let us now assume that the system (1.1) has $l\,(l < 2n)$ independent first integrals

$$F_p(x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n, \mu) = \text{const} \qquad (p = 1, \ldots, l) \quad (2.1)$$

which are analytic in the region of initial conditions of the desired periodic solutions and of the zero value of μ and which are independent of time. Then, following Poincaré [4], the formulas (2.1) can be re-written in the form of differences

$$F_p[x_1(T_0+\alpha), \ldots, x_n(T_0+\alpha), \dot{x}_1(T_0+\alpha), \ldots, \dot{x}_n(T_0+\alpha), \mu] - $$
$$- F_p[x_1(0), \ldots, x_n(0), \dot{x}_1(0), \ldots, \dot{x}_n(0), \mu] = 0$$

which we rewrite, on the basis of (1.2), as

$$F_p[x_1(0)+\psi_1, \ldots, x_n(0)+\psi_n, \dot{x}_1(0)+\psi_{n+1}, \ldots, \dot{x}_n(0)+\psi_{2n}, \mu] - $$
$$- F_p[x_1(0), \ldots, x_n(0), \dot{x}_1(0), \ldots, \dot{x}_n(0), \mu] = 0 \quad (p = 1, \ldots, l) \quad (2.2)$$

Expanding the expressions (2.2) into power series in the ψ variables, we obtain the equations

$$F_p^*(\psi_1, \ldots, \psi_{2n}, \mu) = 0 \quad (2.3)$$

whose left-hand sides are analytic functions of their arguments and vanish when the conditions (1.3) are satisfied. Solving the equations (2.3) for a set of l variables ψ, for example, $\psi_{2n+1-l}, \ldots, \psi_{2n}$, which can always be done [5], provided

$$\left.\frac{D(F_1^*, \ldots, F_l^*)}{D(\psi_{2n+1-l}, \ldots, \psi_{2n})}\right|_{\psi_1=\ldots=\psi_{2n}=0} \neq 0$$

we obtain

$$\psi_{2n+1-p} = \Phi_p(\psi_1, \ldots, \psi_{2n-l}) \qquad (p = 1, \ldots, l)$$

where Φ_p are series expanded in appropriate powers of all the parameters entering into them; these series vanish when

$$\psi_1 = \ldots = \psi_{2n-l} = 0 \tag{2.4}$$

It follows from this that, since the last l equations of the system (1.3) depend on the first $2n - l$ equations, the periodicity conditions

$$\psi_{2n+1-l} = \ldots = \psi_{2n} = 0$$

will be automatically satisfied if the conditions (2.4) are satisfied. Thus, to find a periodic solution of the system (1.1) we need only $2n - l$ of the periodicity conditions (1.3) or $2(2n - 1 - l)$ conditions for the system (1.8)

$$M_q(A_1, \ldots, A_n, B_2, \ldots, B_n) = 0, \qquad N_q(\beta_1, \ldots, \beta_n, \gamma_2, \ldots, \gamma_n, \mu) = 0 \tag{2.5}$$
$$(q = 2, \ldots, 2n - l)$$

The first group of conditions in (2.5) represents $2(2n - 1 - l)$ equations for the basic amplitudes, with $2n - 1$ unknowns A and B. We solve these equations for a set of $2n - 1 - l$ unknowns; this is possible (for example [5]) if the rank of the Jacobian

$$\left\| \begin{matrix} \dfrac{\partial M_2}{\partial A_1} & \cdots & \dfrac{\partial M_2}{\partial A_n} & \dfrac{\partial M_2}{\partial B_2} & \cdots & \dfrac{\partial M_2}{\partial B_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial M_{2n-l}}{\partial A_1} & \cdots & \dfrac{\partial M_{2n-l}}{\partial A_n} & \dfrac{\partial M_{2n-l}}{\partial B_2} & \cdots & \dfrac{\partial M_{2n-l}}{\partial B_n} \end{matrix} \right\| \tag{2.6}$$

is $2n - 1 - l$ in some region of values of the unknowns A and B. We then find that the remaining l of the unknowns A and B may be considered arbitrary parameters in this region.

The second group of conditions represents $2n - 1 - l$ equations which depend on $2n - 1$ unknowns β, γ and the parameter μ. We solve these equations for a set of $2n - 1 - l$ unknowns, expressed in the form of power series in the remaining l unknowns and the parameter μ. From the equations (1.7) it follows that this can always be done [5] if the Jacobian of the matrix (2.6) has a rank of $2n - 1 - l$ in some region of values of the unknowns A and B.

It follows from this that the l free unknowns β and γ may be taken to be arbitrary analytic functions of μ, which vanish at $\mu = 0$. We note that the case in which the matrix (2.6) has a rank less than $2n - 1 - l$ means that the corresponding systems of equations may have multiple roots.

Thus, the periodic solutions of a quasilinear autonomous system with l independent first integrals depends, under certain conditions, on l arbitrary constants and l arbitrary functions of μ. Similar arguments also hold true for quasilinear nonautonomous systems with n degrees of freedom which have $l (l < 2n)$ independent first integrals

$$F_p(x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n, \mu, t) = \text{const} \qquad (p = 1, \ldots, l)$$

which are analytic in the region of initial values of the desired periodic solutions and the zero value of μ and periodic in time with a period 2π.

3. As an example, let us consider a periodic solution of a known [6] equation for the free oscillations of a conservative system with one degree of freedom involving a nonlinear elastic restoring force

$$\ddot{x} + x = \mu x^3 \tag{3.1}$$

which is satisfied by the kinetic energy integral

$$\dot{x}^2 + x^2 - \mu \frac{x^4}{2} = \text{const} \tag{3.2}$$

Choosing the initial conditions of the desired periodic solution in the form

$$x(0) = A + \beta, \qquad \dot{x}(0) = 0 \tag{3.3}$$

and using the notation

$$x(2\pi + \alpha) - x(0) = \psi_1, \qquad \dot{x}(2\pi + \alpha) - \dot{x}(0) = \psi_2 \tag{3.4}$$

we obtain the periodicity conditions

$$\psi_1 = \psi_2 = 0$$

In accordance with Section 2, rewriting the integral (3.2) in the form

$$[\dot{x}(0) + \psi_2]^2 + [x(0) + \psi_1]^2 - \frac{\mu}{2}[x(0) + \psi_1]^4 - \dot{x}^2(0) - x^2(0) + \frac{\mu}{2}x^4(0) = 0$$

we obtain the equation

$$2x(0)[1 - \mu x^2(0)]\psi_1 + 2\dot{x}(0)\psi_2 + \ldots = 0 \tag{3.5}$$

where the unwritten terms involve higher powers of ψ_1 and ψ_2. Substituting the initial values (3.3) into the equations (3.5), we see that, for example, if $A \neq 0$, this equation can be solved for ψ_1

$$\psi_1 = \Phi(\psi_2) \qquad (\Phi(0) = 0)$$

Therefore, the condition $\psi_2 = 0$ is the only independent periodicity condition. From this condition we can find the quantity $\alpha = \alpha(\beta, \mu)$ if $A \neq 0$ [7].

Thus, the periodic solution of the equation (3.1) depends on one arbitrary non-zero constant and one arbitrary function of μ. This solution can be found, for example, by formula (1.10) of the reference quoted.

BIBLIOGRAPHY

1. Proskuriakov, A.P., Ob odnom svoistve periodicheskikh reshenii kvazilineinykh avtonomnyk sistem s neskol'kimi stepeniami svobody (On a property of periodic solutions of quasilinear autonomous systems with several degrees of freedom). *PMM* Vol. 24, No. 4, 1960.

2. Proskuriakov, A.P., K postroeniiu periodicheskikh reshenii kvazilineinykh avtonomnykh sistem s neskol'kimi stepeniami svobody (On the construction of periodic solutions of quasilinear autonomous systems with several degrees of freedom). *PMM* Vol. 26, No. 2, 1962.

3. Proskuriakov, A.P., Periodicheskie kolebaniia kvazilineinoi avtonomnoi sistemy s dvumia stepeniami svobody (Periodic oscillations of a quasilinear autonomous system with two degrees of freedom). *PMM* Vol. 24, No. 6, 1960.

4. Poincaré, H., *Les méthodes nouvelles de la mécanique céleste.* Vol. 1, Ch. 3. Paris, 1892.

5. Goursat, E., *Kurs matematicheskogo analiza (A Course of Mathematical Analysis).* Vol. 1, Ch. 2. GTTI, 1933.

6. Malkin, I.G., *Nekotorye zadachi teorii nelineinykh kolebanii (Some Problems in the Theory of Nonlinear Oscillations).* Gostekhizdat, 1956.

7. Proskuriakov, A.P., Postroenie periodicheskik reshenii avtonomnykh sistem s odnoi stepen'iu svobody v sluchae proizvol'nykh veshchestvennykh kornei uravneniia osnovnykh amplitud (Construction of periodic solutions of autonomous systems with one degree of freedom in the case of arbitrary real roots of the equation for the basic amplitudes). *PMM* Vol. 22, No. 4, 1958.

Translated by A.S.